QUANTUM CURRENTS REALIZATION OF THE ELLIPTIC QUANTUM GROUPS $E_{\tau,\eta}(\mathfrak{sl}_2)$

B. ENRIQUEZ

Abstract. We review the construction by G. Felder and the author of the realization of the elliptic quantum groups by quantum currents.

. The elliptic quantum groups were introduced by G. Felder in [14]. These are algebraic objects based on a solution $R(z,\lambda)$ of the dynamical Yang-Baxter equation. Here dynamical means that in addition to the spectral parameter z, the R-matrix depends on a parameter λ, which belongs to a product of elliptic curves, and that these parameters undergo shifts in the various terms of the equation.

The aim of this paper is to review the construction by G. Felder and the author ([9]) of the realization of elliptic quantum groups $E_{\tau,\eta}(\mathfrak{sl}_2)$ by quantum current algebras. This construction relies on quasi-Hopf algebra techniques. We introduce (sect. 2) a quantum loop algebra $U_\hbar\mathfrak{g}(\tau)$ (τ is the elliptic parameter, $\mathfrak{g}=\mathfrak{sl}_2$) that presents analogies with $E_{\tau,\eta}(\mathfrak{sl}_2)$. Namely, it has the property that the image in representations of its classical r-matrix coincides with the classical limit of $R(z,\lambda)$. $U_\hbar\mathfrak{g}(\tau)$ is endowed with "Drinfeld-type" coproducts Δ and $\bar\Delta$ (see [6]), conjugated by a twist F (sect. 2.3). Then our goal is to construct, in this algebra, a solution of the DYBE yielding $R(z,\lambda)$ in finite-dimensional representations. For that, we make use of a result of O. Babelon, D. Bernard and E. Billey (BBB). This result extends to the dynamical situation the theory of Drinfeld twists (see [7]): it states that a solution of the so-called twisted Hopf cocycle equation, in some quasi-triangular Hopf algebra, yields a solution of the DYBE at the algebra level. To construct such a solution, we solve a factorization problem for the twist F (sect. 4.3). This factorization in turn relies on some results on Hopf algebra pairings within the quantum loop algebra. After we have obtained a DYBE solution in $U_\hbar\mathfrak{g}(\tau)^{\otimes 2}$, we study its representations and construct from it L-operators, that satisfy RLL relations which are exactly the elliptic quantum groups relations (sect. 4.6).

Date: August 1997.

To explain the strategy we are following, we first review (sect. 4.1) the treatment of [8] of the rational analogue of our elliptic situation. In that case, the solution of a factorization problem allows to construct a twist conjugating the Drinfeld coproduct for Yangians with the usual coproduct. Another derivation of this twist was earlier obtained by S. Khoroshkin and V. Tolstoy ([21]).

The interest of this construction lies in that it permits to embed the elliptic quantum group in an algebra "with central extension". This allows to apply quantum Kac-Moody algebra techniques to the study of this algebra. For example, one may expect that the elliptic quantum KZB equations of [15] are obtained in terms of intertwiners or coinvariants from the algebras presented here (such a study should be related to the apporach of [1, 2] to the Ruijsenaars-Schneiders models). Another interesting application would be the study of the center of the algebra $U_\hbar\mathfrak{g}(\tau)$ at the critical level, as a Poisson algebra, and its possible deformation as some kind of $\mathcal{W}$-algebra, in the spirit of [18]. Another subject of possible interest would be to study the connection with other types of elliptic quantum groups – those arising from the Belavin-Baxter solution or those studied in [19]. Finally, it would be interesting to find relations satisfied by a finite set of generators of the quantum loop algebra $U_\hbar\mathfrak{g}(\tau)$, or its subalgebra $U_\hbar\mathfrak{g}_{\mathcal{O}}$ (see 4.1) analogous to the Drinfeld presentation for Yangians in terms of generators $I(a)$ and $J(a)$.

1. Elliptic quantum groups

1.1. **Function spaces associated with elliptic curves.** Let τ be a complex number of positive imaginary part, and E be the elliptic curve $\mathbb{C}/\Gamma$, where $\Gamma = \mathbb{Z} + \tau\mathbb{Z}$. The basic theta-function θ associated to E is defined by the conditions this it is holomorphic on $\mathbb{C}$, $\theta'(0) = 1$, the only zeroes of θ are the points of Γ, $\theta(z+1) = -\theta(z)$, and $\theta(z+\tau) = -e^{-i\pi\tau}e^{-2i\pi z}\theta(z)$. θ is then odd.

For $\lambda \in \mathbb{C}$, define L_λ as follows. If λ does not belong to Γ, define L_λ to be the set of holomorphic functions on $\mathbb{C} - \Gamma$, 1-periodic and such that $f(z+\tau) = e^{-2i\pi\lambda}f(z)$. For $\lambda = 0$, define L_λ as the maximal isotropic subspace of $\mathbb{C}((z))$ (for the pairing $\langle f, g\rangle = \mathrm{res}_0(fgdz)$) containing all holomorphic functions f on $\mathbb{C} - \Gamma$, Γ-periodic, such that $\oint_a f(z)dz = 0$, where a is the cycle $(i\epsilon, i\epsilon + 1)$ (with ϵ small and > 0). Finally, define $L_\lambda = e^{-2i\pi mz}L_0$ for $\lambda = n + m\tau$.

We then have

$$L_\lambda = \oplus_{j\geq 0}\mathbb{C}\left(\frac{\theta'}{\theta}\right)^{(j)} e^{-2i\pi mz}, \quad \text{if} \quad \lambda = n + m\tau, \tag{1}$$

$$L_\lambda = \oplus_{i\geq 0}\mathbb{C}\left(\frac{\theta(\lambda+z)}{\theta(z)}\right)^{(i)}, \quad \text{if} \quad \lambda \in \mathbb{C}-\Gamma, \tag{2}$$

where we let $g' = \partial_z g$ and $g^{(i)} = \partial_z^i g$.

We will set

$$e_{i,\lambda}(z) = \left(\frac{\theta(\lambda+z)}{\theta(z)}\right)^{(i)}. \tag{3}$$

1.2. **Elliptic R-matrix.** Let (v_1, v_{-1}) be the standard basis of $\mathbb{C}^2$ and E_{ij} the endomorphism of $\mathbb{C}^2$ defined by $E_{ij}v_\alpha = \delta_{j\alpha}v_i$. Let $\hbar$ be a formal variable and set

$$R(z,\lambda) = E_{11}\otimes E_{11} + E_{-1,-1}\otimes E_{-1,-1} + \frac{\theta(z)}{\theta(z+\hbar)}\frac{\theta(\lambda+\hbar)\theta(\lambda-\hbar)}{\theta(\lambda)^2}E_{1,1}\otimes E_{-1,-1} \tag{4}$$
$$+\frac{\theta(z)}{\theta(z+\hbar)}E_{-1,-1}\otimes E_{11} + \frac{\theta(z+\lambda)\theta(\hbar)}{\theta(z+\hbar)\theta(\lambda)}E_{1,-1}\otimes E_{-1,1}$$
$$-\frac{\theta(z-\lambda)\theta(\hbar)}{\theta(z+\hbar)\theta(\lambda)}E_{-1,1}\otimes E_{1,-1}.$$

1.3. **The dynamical Yang-Baxter equation.** This matrix satisfies the equation

$$R^{(12)}(z_{12},\lambda)R^{(13)}(z_{13},\lambda+\hbar\bar{h}^{(2)})R^{(23)}(z_{23},\lambda) \tag{5}$$
$$= R^{(23)}(z_{23},\lambda+\hbar\bar{h}^{(1)})R^{(13)}(z_{13},\lambda)R^{(12)}(z_{12},\lambda+\hbar\bar{h}^{(3)}),$$

where $z_{ij} = z_i - z_j$ and $z_i, i = 1,2,3$ are generic complex numbers. We set $\bar{h} = E_{11} - E_{22}$, and $\bar{h}^{(k)}$ is the image of $\bar{h}$ in the kth factor of $\mathrm{End}(\mathbb{C}^2)^{\otimes 3}$. For i,j,k a permutation of $1,2,3$, and any $v \in (\mathbb{C}^2)^{\otimes 3}$ such that $\bar{h}^{(k)}v = \mu v$, we set $R^{(ij)}(z,\lambda+\hbar h^{(k)})v = R^{(ij)}(z,\lambda+\hbar\mu)v$.

Equation (5) is called the dynamical Yang-Baxter equation.

1.4. **Elliptic quantum groups.** Set $\eta = \hbar/2$. The elliptic quantum group $E_{\tau,\eta}(\mathfrak{sl}_2)$ is defined as the algebra generated by h and the $a_i(\lambda), b_i(\lambda), c_i(\lambda), d_i(\lambda), i\geq 0, \lambda\in\mathbb{C}-\Gamma$, subject to the relations

$$[h, a_i(\lambda)] = [h, d_i(\lambda)] = 0, \quad [h, b_i(\lambda)] = -2b_i(\lambda), \quad [h, c_i(\lambda)] = 2c_i(\lambda),$$

and if we set

$$a(z,\lambda) = \sum_{i\geq 0} a_i(\lambda)e_{i,+\hbar h/2}(z), \quad b(z,\lambda) = \sum_{i\geq 0} b_i(\lambda)e_{i,\lambda+\hbar(h-2)/2}(z),$$

$$c(z,\lambda) = \sum_{i\geq 0} c_i(\lambda)e_{i,-\lambda-\hbar(h+2)/2}(z), \quad d(z,\lambda) = \sum_{i\geq 0} d_i(\lambda)e_{i,-\hbar h/2}(z),$$

and

$$L(z,\lambda) = \begin{pmatrix} a(z,\lambda) & b(z,\lambda) \\ c(z,\lambda) & d(z,\lambda) \end{pmatrix}, \tag{6}$$

the relations

$$\begin{aligned} R^{(12)}(z_1 - z_2, \lambda + \hbar h) L^{(1)}(z_1,\lambda) L^{(2)}(z_2, \lambda + \hbar h^{(1)}) \\ = L^{(2)}(z_2,\lambda) L^{(1)}(z_1, \lambda + \hbar h^{(2)}) R^{(12)}(\lambda, z_1 - z_2) \end{aligned} \tag{7}$$

and

$$\mathrm{Det}(z,\lambda) = d(z-\hbar,\lambda)a(z,\lambda-\hbar) - b(z-\hbar,\lambda)c(z,\lambda-\hbar)\frac{\theta(\lambda+\hbar h+\hbar)}{\theta(\lambda+\hbar h)} = 1. \tag{8}$$

(These relations are made explicit in [16].)

We use the convention that $x_{\lambda+\hbar h}(z) = \sum_{i\geq 0} \partial_\lambda^i x_\lambda(z)(\hbar h)^i/i!$.

1.5. **Another presentation.**

The formulas defining the quantum groups of [14] are based on an R-matrix $\bar{R}$ slightly different from R. Let φ be a solution to the equation

$$\frac{\varphi(\lambda-\hbar)}{\varphi(\lambda+\hbar)} = \frac{\theta(\lambda)}{\theta(\lambda+\hbar)};$$

then we have

$$\bar{R}(z,\lambda) = \varphi(\lambda + \hbar h^{(2)}) R(z,\lambda) \varphi(\lambda + \hbar h^{(1)})^{-1}.$$

The L-matrix of the elliptic quantum group based on $\bar{R}$ can be connected with that of $E_{\tau,\eta}(\mathfrak{sl}_2)$ using the transformation

$$\bar{L}(z,\lambda) = \varphi(\lambda+\hbar h) L(z,\lambda) \varphi(\lambda + \hbar h^{(1)})^{-1}.$$

2. Quantum currents algebra

The quantum currents algebra used in [9] is an example of a family of algebras which were introduced in [11]. These algebras are associated to the data of an algebraic curve and a rational differential. In [12], it was shown that these algebras can be endowed with quasi-Hopf structures, quantizing certain Manin pairs.

2.1. **Classical structures.**

2.1.1. *Manin pairs.* Let $\mathcal{K} = \mathbb{C}((z))$ be the completed local field of E at its origin 0, and $\mathcal{O} = \mathbb{C}[[z]]$ the completed local ring at the same point. Endow $\mathcal{K}$ with the scalar product $\langle , \rangle_{\mathcal{K}}$ defined by

$$\langle f, g\rangle_{\mathcal{K}} = \mathrm{res}_0(fgdz).$$

Define on $\mathcal{K}$ the derivation ∂ to be equal to d/dz. Then ∂ is invariant w.r.t $\langle , \rangle_{\mathcal{K}}$, and $\mathcal{O}$ is a maximal isotropic subring of $\mathcal{K}$.

Let us set $\mathfrak{a} = \mathfrak{sl}_2(\mathbb{C})$, and denote by $\langle , \rangle_{\mathfrak{a}}$ an invariant scalar product on $\mathfrak{a}$. Let us set $\mathfrak{g} = (\mathfrak{a} \otimes \mathcal{K}) \oplus \mathbb{C}D \oplus \mathbb{C}K$; let us define on $\mathfrak{g}$ the Lie algebra stucture defined by the central extension of $\mathfrak{a} \otimes \mathcal{K}$

$$c(x \otimes f, y \otimes g) = \langle x, y\rangle_{\mathfrak{a}} \langle f, \partial g\rangle_{\mathcal{K}} K$$

and by the derivation $[D, x \otimes f] = x \otimes \partial f$.

Let $\mathfrak{g}_{\mathcal{O}}$ be the Lie subalgebra of $\mathfrak{g}$ equal to $(\mathfrak{a} \otimes \mathcal{O}) \oplus \mathbb{C}D$. Define $\langle , \rangle_{\mathfrak{a}\otimes\mathcal{K}}$ as the tensor product of $\langle , \rangle_{\mathfrak{a}}$ and $\langle , \rangle_{\mathcal{K}}$, and $\langle , \rangle_{\mathfrak{g}}$ as the scalar product on $\mathfrak{g}$ defined by $\langle , \rangle_{\mathfrak{g}}|_{\mathfrak{a}\otimes\mathcal{K}} = \langle , \rangle_{\mathfrak{a}\otimes\mathcal{K}}$, $\langle D, \mathfrak{a} \otimes \mathcal{K}\rangle_{\mathfrak{g}} = \langle K, \mathfrak{a} \otimes \mathcal{K}\rangle_{\mathfrak{g}} = 0$, and $\langle D, K\rangle_{\mathfrak{g}} = 1$. Then $\mathfrak{g}_{\mathcal{O}}$ is a maximal isotropic Lie subalgebra of $\mathfrak{g}$.

A maximal isotropic supplementary to $\mathfrak{g}_{\mathcal{O}}$ on $\mathfrak{g}$ is defined by

$$\mathfrak{g}_\lambda = (\mathfrak{h} \otimes L_0) \oplus (\mathfrak{n}_+ \otimes L_\lambda) \oplus (\mathfrak{n}_- \otimes L_{-\lambda}) \oplus \mathbb{C}K. \tag{9}$$

Therefore,

$$\mathfrak{g} = \mathfrak{g}_{\mathcal{O}} \oplus \mathfrak{g}_\lambda \tag{10}$$

define a Lie quasi-bialgebra structure on $\mathfrak{g}_{\mathcal{O}}$, and (as in [12]), of double Lie quasi-bialgebra on $\mathfrak{g}$. Its classical r-matrix is given by the formula

$$r_\lambda = D \otimes K + \sum_i \frac{1}{2} h[z^i/i!] \otimes h[e_{i;0}] + e[z^i/i!] \otimes f[e_{i;\lambda}] + f[z^i/i!] \otimes e[e_{i;-\lambda}],$$

because $(z^i/i!)_{i\geq 0}, (e_{i;\lambda})_{i\geq 0}$ are dual bases of $\mathcal{O}$ and L_λ; we denote $x \otimes f$ by $x[f]$; in other terms,

$$r_\lambda(z, w) = \frac{1}{2}(h \otimes h)\frac{\theta'}{\theta}(z - w) + (e \otimes f)\frac{\theta(z - w + \lambda)}{\theta(z - w)\theta(\lambda)} + (f \otimes e)\frac{\theta(z - w - \lambda)}{\theta(z - w)\theta(-\lambda)} \tag{11}$$
$$+ D \otimes K.$$

In what follows, we will set

$$e^i = z^i/i!. \tag{12}$$

Remark 1. The expansion of $R(z-w, \lambda)$ in powers of $\hbar$ is $(\pi_z \otimes \pi_w)(1+ \hbar r_\lambda + \cdots)$, where π_z and π_w are the 2-dimensional evaluation represntations of $\mathfrak{g}$ at the points z and w. This may be viewed as an indication that the quantization of the Manin pair (10) yields a realization of the elliptic quantum group relations.

Remark 2. The role played by $r_\lambda(z,w)$ in the elliptic KZB equations and the Hitchin systems are explained in [17] and [10].

2.1.2. *Manin triples.* It is useful to consider the following twists of the Lie quasi-bialgebra structures provided by (10).

Let us set

$$\mathfrak{g}_+ = (\mathfrak{n}_+ \otimes \mathcal{K}) \oplus (\mathfrak{h} \otimes L_0) \oplus \mathbb{C}K, \quad \mathfrak{g}_- = (\mathfrak{n}_- \otimes \mathcal{K}) \oplus (\mathfrak{h} \otimes L_0) \oplus \mathbb{C}D, \tag{13}$$

and

$$\bar{\mathfrak{g}}_+ = (\mathfrak{n}_- \otimes \mathcal{K}) \oplus (\mathfrak{h} \otimes L_0) \oplus \mathbb{C}K, \quad \bar{\mathfrak{g}}_- = (\mathfrak{n}_+ \otimes \mathcal{K}) \oplus (\mathfrak{h} \otimes L_0) \oplus \mathbb{C}D. \tag{14}$$

Then the decompositions

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-, \quad \mathfrak{g} = \bar{\mathfrak{g}}_+ \oplus \bar{\mathfrak{g}}_- \tag{15}$$

are decompositions of $\mathfrak{g}$ as direct sums of isotropic subalgebras. This defines two Lie bialgebra structures on $\mathfrak{g}$ Manin triples; both are connected by a twist. Also, there is some twist connecting them with the Lie quasi-bialgebra structure (10). The problem which we are going to solve is to quantize these structures.

2.2. **Quantum algebra.** We now present the algebra $U_\hbar\mathfrak{g}(\tau)$ deforming the enveloping algebra of $\mathfrak{g}$. We will also denote this algebra by $A(\tau)$. Generators of $U_\hbar\mathfrak{g}(\tau)$ are D, K and the $x[\epsilon]$, $x = e, f, h$, $\epsilon \in \mathcal{K}$; they are subject to the relations

$$x[\alpha\epsilon] = \alpha x[\epsilon], \quad x[\epsilon + \epsilon'] = x[\epsilon] + x[\epsilon'], \quad \alpha \in \mathbb{C}, \epsilon, \epsilon' \in \mathcal{K}.$$

They serve to define the generating series

$$x(z) = \sum_{i \in \mathbb{Z}} x[\epsilon^i]\epsilon_i(z), \quad x = e, f, h,$$

$(\epsilon^i)_{i\in\mathbb{Z}}, (\epsilon_i)_{i\in\mathbb{Z}}$ dual bases of $\mathcal{K}$; recall that $(e^i)_{i\in\mathbb{N}}, (e_{i;0})_{i\in\mathbb{N}}$ are dual bases of $\mathcal{O}$ and L_0 and set

$$h^+(z) = \sum_{i\in\mathbb{N}} h[e^i]e_{i;0}(z), \quad h^-(z) = \sum_{i\in\mathbb{N}} h[e_{i;0}]e^i(z).$$

We will also use the series

$$K^+(z) = e^{(\frac{q^\partial - q^{-\partial}}{2\partial}h^+)(z)}, \quad K^-(z) = q^{h^-(z)},$$

where $q = e^\hbar$. The relations presenting $U_\hbar\mathfrak{g}(\tau)$ are then

$$K \text{ is central}, \quad [K^+(z), K^+(w)] = [K^-(z), K^-(w)] = 0, \tag{16}$$

$$\theta(z-w-\hbar)\theta(z-w+\hbar+\hbar K)K^+(z)K^-(w) \tag{17}$$
$$=\theta(z-w+\hbar)\theta(z-w-\hbar+\hbar K)K^-(w)K^+(z),$$

$$K^+(z)e(w)K^+(z)^{-1}=\frac{\theta(z-w+\hbar)}{\theta(z-w-\hbar)}e(w) \tag{18}$$

$$K^-(z)e(w)K^-(z)^{-1}=\frac{\theta(w-z+\hbar K+\hbar)}{\theta(w-z+\hbar K-\hbar)}e(w), \tag{19}$$

$$K^+(z)f(w)K^+(z)^{-1}=\frac{\theta(w-z+\hbar)}{\theta(w-z-\hbar)}f(w), K^-(z)f(w)K^-(z)^{-1}=\frac{\theta(z-w+\hbar)}{\theta(z-w-\hbar)}f(w), \tag{20}$$

$$\theta(z-w-\hbar)e(z)e(w)=\theta(z-w+\hbar)e(w)e(z), \tag{21}$$

$$\theta(w-z-\hbar)f(z)f(w)=\theta(w-z+\hbar)f(w)f(z), \tag{22}$$

$$[e(z),f(w)]=\frac{1}{\hbar}\left(\delta(z,w)K^+(z)-\delta(z,w-\hbar K)K^-(w)^{-1}\right). \tag{23}$$

Here δ denotes, as usual, the formal series $\sum_{i\in\mathbb{Z}}z^iw^{-i-1}$.

Similar relations were presented in [5].

2.3. **Coproducts.** The algebra $U_\hbar\mathfrak{g}(\tau)$ is endowed with a Hopf structure given by the coproduct Δ defined by

$$\Delta(K^+(z))=K^+(z)\otimes K^+(z),\quad \Delta(K^-(z))=K^-(z)\otimes K^-(z+\hbar K_1), \tag{24}$$

$$\Delta(e(z))=e(z)\otimes K^+(z)+1\otimes e(z), \tag{25}$$

$$\Delta(f(z))=f(z)\otimes 1+K^-(z)^{-1}\otimes f(z+\hbar K_1), \tag{26}$$

$$\Delta(D)=D\otimes 1+1\otimes D,\quad \Delta(K)=K\otimes 1+1\otimes K, \tag{27}$$

the counit ε, and the antipode S defined by them; we set $K_1=K\otimes 1, K_2=1\otimes K$.

$U_\hbar\mathfrak{g}(\tau)$ is also endowed with another Hopf structure given by the coproduct $\bar\Delta$ defined by

$$\bar\Delta(K^+(z))=K^+(z)\otimes K^+(z),\quad \bar\Delta(K^-(z))=K^-(z)\otimes K^-(z+\hbar K_1), \tag{28}$$

$$\bar\Delta(e(z))=e(z-\hbar K_2)\otimes K^-(z-\hbar K_2)^{-1}+1\otimes e(z), \tag{29}$$

$$\bar\Delta(f(z))=f(z)\otimes 1+K^+(z)\otimes f(z), \tag{30}$$

$$\bar{\Delta}(D) = D \otimes 1 + 1 \otimes D, \quad \bar{\Delta}(K) = K \otimes 1 + 1 \otimes K, \tag{31}$$

the counit ε, and the antipode $\bar{S}$ defined by them.

The Hopf structures associated with Δ and $\bar{\Delta}$ are connected by a twist

$$F = \exp\left(\hbar \sum_{i \in \mathbb{Z}} e[\epsilon_i] \otimes f[\epsilon^i]\right), \tag{32}$$

where $(\epsilon^i)_{\in \mathbb{Z}}$ is the basis of $\mathcal{K}$ dual to $(\epsilon_i)_{i\in\mathbb{Z}}$ w.r.t. $\langle , \rangle_{\mathcal{K}}$; that is, we have $\bar{\Delta} = \mathrm{Ad}(F) \circ \Delta$.

Then F satisfies the cocycle equation

$$(F \otimes 1)(\Delta \otimes 1)(F) = (1 \otimes F)(1 \otimes \Delta)(F) \tag{33}$$

(see [12], Prop. 3.1).

Proposition 2.1. *$(U_\hbar\mathfrak{g}(\tau), \Delta)$ and $(U_\hbar\mathfrak{g}(\tau), \bar{\Delta})$ are quantizations of the Manin triple structures defined by (15). The universal R-matrix of $(U_\hbar\mathfrak{g}(\tau), \Delta)$ is*

$$\mathcal{R}_\infty = q^{D\otimes K} q^{\frac{1}{2}\sum_{i\geq 0} h[e^i]\otimes h[e_{i;0}]} q^{\sum_{i\in\mathbb{Z}} e[\epsilon^i]\otimes f[\epsilon_i]},$$

and for $(U_\hbar\mathfrak{g}(\tau), \bar{\Delta})$ it is $F^{(21)}\mathcal{R}_\infty F^{-1}$.

3. The realization

3.1. **Half-currents.** Fix a complex number λ and set for $x = e, f, K^+$,

$$x^+_\lambda(z) = \sum_i x[e^i] e_{i;\lambda}(z), \tag{34}$$

and for $x = e, f, K^-$,

$$x^-_\lambda(z) = \sum_i x[e_{i;-\lambda}] e^i(z); \tag{35}$$

recall that $(e^i), (e_{i;\lambda})$ are dual bases of $\mathcal{O}$ and L_λ.

The fields $e(z)$ and $f(z)$ are then split according to

$$e(z) = e^+_\lambda(z) + e^-_\lambda(z), \quad f(z) = f^+_{-\lambda}(z) + f^-_{-\lambda}(z); \tag{36}$$

we call the expression $x^\pm_\lambda(z)$ "half-currents". Let us introduce the generating series $k^+(z)$ and $k^-(z)$, defined by

$$k^+(z) = e^{(\frac{q^\partial - 1}{2\partial} h^+)(z)}, \quad k^-(z) = q^{(\frac{1}{1+q^{-\partial}} h^-)(z)}; \tag{37}$$

they satisfy the relations

$$K^+(z) = k^+(z) k^+(z-\hbar), \quad K^-(z) = k^-(z) k^-(z-\hbar). \tag{38}$$

3.2. **Realization.** Introduce the L-operators

$$L_\lambda^+(\zeta) = \begin{pmatrix} 1 & \theta(\hbar) f^+_{\lambda+\hbar h-\hbar}(\zeta) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} k^+(z-\hbar) & 0 \\ 0 & k^+(z)^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \hbar e^+_{-\lambda}(\zeta) & 1 \end{pmatrix}, \tag{39}$$

$$L_\lambda^-(\zeta) = \begin{pmatrix} 1 & 0 \\ \hbar e^-_{-\lambda}(\zeta - K\hbar) & 1 \end{pmatrix} \begin{pmatrix} k^-(\zeta-\hbar) & 0 \\ 0 & k^-(\zeta)^{-1} \end{pmatrix} \begin{pmatrix} 1 & \theta(\hbar) f^-_{\lambda+\hbar h-\hbar}(\zeta) \\ 0 & 1 \end{pmatrix}. \tag{40}$$

The main result of this paper is:

Theorem 3.1. *Set*

$$\begin{aligned} R^-(z,\lambda) = {} & E_{11}\otimes E_{11} + E_{-1,-1}\otimes E_{-1,-1} + \frac{\theta(z)}{\theta(z-\hbar)} E_{11}\otimes E_{-1,-1} \\ & + \frac{\theta(\lambda+\hbar)\theta(\lambda-\hbar)}{\theta(\lambda)^2}\frac{\theta(z)}{\theta(z-\hbar)} E_{-1,-1}\otimes E_{11} - \frac{\theta(z+\lambda)\theta(\hbar)}{\theta(z-\hbar)\theta(\lambda)} E_{1,-1}\otimes E_{-1,1} \\ & + \frac{\theta(z-\lambda)\theta(\hbar)}{\theta(z-\hbar)\theta(\lambda)} E_{-1,1}\otimes E_{1,-1}, \end{aligned} \tag{41}$$

and $R^+(z,\lambda) = R(z,\lambda)$. The L-operators $L^\pm_\lambda(\zeta)$ satisfy the relations

$$R^\pm(\zeta-\zeta',\lambda) L^{\pm(1)}_{\lambda+\hbar h^{(2)}}(\zeta) L^{\pm(2)}_\lambda(\zeta') = L^{\pm(2)}_{\lambda+\hbar h^{(1)}}(\zeta') L^{\pm(1)}_\lambda(\zeta) R^\pm(\zeta-\zeta',\lambda+\hbar h) \tag{42}$$

$$\begin{aligned} & L^{-(1)}_\lambda(\zeta) R^-(\zeta-\zeta',\lambda+\hbar h) L^{+(2)}_\lambda(\zeta') \\ & \quad = L^{+(2)}_{\lambda+\hbar h^{(1)}}(\zeta') R^-(\zeta-\zeta'-K\hbar,\lambda) L^{-(1)}_{\lambda+\hbar h^{(2)}}(\zeta) \frac{A(\zeta,\zeta'+K\hbar)}{A(\zeta,\zeta')}, \end{aligned} \tag{43}$$

where

$$A(\zeta,\zeta') = \exp(\sum_{i\geq 0} \left(\frac{1}{\partial}\frac{q^\partial - 1}{q^\partial + 1} e^i\right)(\zeta) e_{i;0}(\zeta')). \tag{44}$$

This result can be viewed as an elliptic analogue of the result of [4].

4. About the proof

4.1. **A rational analogue.** In [8], we constructed a Hopf algebra cocycle in the Yangian double $A = DY(\mathfrak{sl}_2)$, conjugating Drinfeld's coproduct to the usual one. $DY(\mathfrak{sl}_2)$ is a rational version of $U_\hbar\mathfrak{g}(\tau)$ (it was first introduced in [20]). This means that it has the same presentation as $U_\hbar\mathfrak{g}(\tau)$, replacing the theta-functions by their arguments.

For $x = e, f, h$, let x_n be the analogue of $x[z^n]$. $DY(\mathfrak{sl}_2)$ contains two "negative modes" and "nonnegative modes" subalgebras $A^{<0}$ and $A^{\geq 0}$, repectively generated by K and the $x_n, n < 0$ and D and the $x_n, n \geq 0$, $x = e, f, h$.

The Yangian coproduct Δ_{Yg} on $DY(\mathfrak{sl}_2)$ defines a Hopf algebra structure on $DY(\mathfrak{sl}_2)$, for which $A^{\geq 0}$ and $A^{<0}$ are both Hopf subalgebras. On the other hand, the rational analogues Δ_{rat} and $\bar{\Delta}_{\mathrm{rat}}$ of Δ and $\bar{\Delta}$ have the following properties:

$$\Delta_{\mathrm{rat}}(A^{\geq 0}) \subset A \otimes A^{\geq 0}, \quad \Delta_{\mathrm{rat}}(A^{<0}) \subset A^{<0} \otimes A,$$
$$\bar{\Delta}_{\mathrm{rat}}(A^{\geq 0}) \subset A^{\geq 0} \otimes A, \quad \bar{\Delta}_{\mathrm{rat}}(A^{<0}) \subset A \otimes A^{<0}.$$

Based on the study of Hopf algebra duality within $DY(\mathfrak{sl}_2)$, we show that we have a decomposition of the rational analogue of F, F_{rat}, as a product $F_{\mathrm{rat}} = F_2 F_1$, with $F_1 \in A^{<0} \otimes A^{\geq 0}$ and $F_2 \in A^{\geq 0} \otimes A^{<0}$.

Then the twist $\Delta = F_1 \Delta_{\mathrm{rat}} F_1^{-1}$ coincides with the conjugation $F_2^{-1} \bar{\Delta}_{\mathrm{rat}} F_2$. It follows that Δ satisfies both

$$\Delta(A^{\geq 0}) \subset A^{\geq 0} \otimes A^{\geq 0} \quad \text{and} \quad \Delta(A^{<0}) \subset A^{<0} \otimes A^{<0}.$$

On the other hand, Δ defines a Hopf algebra structure on $DY(\mathfrak{sl}_2)$. Indeed, the associator corresponding to F_1 is expressed as

$$\Phi = F_1^{(12)} (\Delta \otimes 1)(F_1) \left(F_1^{(23)} (1 \otimes \Delta)(F_1) \right)^{-1}.$$

We have clearly $\Phi \in A^{<0} \otimes A \otimes A^{\geq 0}$. Since we also have

$$\Phi = \left((\bar{\Delta} \otimes 1)(F_2) F_2^{(12)} \right)^{-1} (1 \otimes \bar{\Delta})(F_2) F_2^{(23)},$$

we also see that $\Phi \in A^{\geq 0} \otimes A \otimes A^{<0}$. Therefore, $\Phi = 1 \otimes a \otimes 1$, for a certain $a \in A$. On the other hand, as Φ is obtained by twisting a quasi-Hopf structure, it should satisfy the compatibility condition (see [7])

$$(\Delta \otimes id \otimes id)(\Phi)(id \otimes id \otimes \Delta)(\Phi) = (\Phi \otimes 1)(id \otimes \Delta \otimes id)(\Phi)(1 \otimes \Phi),$$

so that $a = 1$. Therefore, $\Phi = 1$ and F_1 satisfies the Hopf cocycle equation. It follows that Δ defines Hopf algebra structure on $DY(\mathfrak{sl}_2)$, which we can show to coincide with Δ_{Yg} (see [8]).

The strategy followed in [9] can be described as follows. $U_\hbar\mathfrak{g}(\tau)$ contains an analogue of $A^{\geq 0}$, its subalgebra $U_\hbar\mathfrak{g}_{\mathcal{O}}$ generated by the $x[\epsilon], \epsilon \in \mathcal{O}, x = e, f, h$. It contains no analogue of $A^{<0}$. However, there are certain subalgebras of $\mathrm{Hol}(\mathbb{C} - \Gamma, U_\hbar\mathfrak{g}(\tau)^{\otimes 2})$ and $\mathrm{Hol}(\mathbb{C} - \Gamma, U_\hbar\mathfrak{g}(\tau)^{\otimes 3})$, playing the roles of $A^{\geq 0} \otimes A^{<0}$ and $A^{<0} \otimes A^{\geq 0}$, respectively $A^{\geq 0, <0} \otimes A^{\otimes 2}$ and $A^{\otimes 2} \otimes A^{\geq 0, <0}$. We will decompose F as a product of elements of these algebras. This will give rise to a solution of the twisted cocycle equation, and by BBB, to a solution of the DYBE in $\mathrm{Hol}(\mathbb{C} - \Gamma, U_\hbar\mathfrak{g}(\tau)^{\otimes 2})$. This solution will happen to coincide with $R(z, \lambda)$ in suitable representations. Therefore, the corresponding L-operators will satisfy the elliptic quantum group relations. Let us see now in more detail how this program is realized.

4.2. **Subalgebras of $Hol(\mathbb{C}-\Gamma, U_\hbar\mathfrak{g}(\tau)^{\otimes 2})$ and $Hol(\mathbb{C}-\Gamma, U_\hbar\mathfrak{g}(\tau)^{\otimes 3})$.** For X a vector space, call $\mathrm{Hol}(\mathbb{C}-\Gamma, X)$ the space of holomorphic functions from $\mathbb{C}-\Gamma$ to X and set $\mathrm{Hol}(\mathbb{C}-\Gamma) = \mathrm{Hol}(\mathbb{C}-\Gamma,\mathbb{C})$. The parameter in $\mathbb{C}-\Gamma$ will be identified with the spectral parameter λ.

Definition 4.1. *Let us define A^{-+} to be the subalgebra of* $\mathrm{Hol}(\mathbb{C}-\Gamma, A(\tau)^{\otimes 2})$ *generated (over* $\mathrm{Hol}(\mathbb{C}-\Gamma)$*) by $h^{(2)}$ and the $e^{-(1)}_{-\lambda-\hbar h^{(2)}}[\epsilon] f^{(2)}[r]$, with $\epsilon \in \mathcal{K}$ and $r \in \mathcal{O}$, and A^{+-} as the subalgebra of* $\mathrm{Hol}(\mathbb{C}-\Gamma, A(\tau)^{\otimes 2})$ *generated (over* $\mathrm{Hol}(\mathbb{C}-\Gamma)$*) by $h^{(2)}$ and the $e^{(1)}[r] f^{-(2)}_{\lambda+\hbar h^{(2)}-2\hbar}[\epsilon]$, with $r \in \mathcal{O}$, $\epsilon \in \mathcal{K}$.*

Definition 4.2. *Let us define $A^{-,\cdot,\cdot}$ as the subspace of the algebra* $\mathrm{Hol}(\mathbb{C}-\Gamma, A(\tau)^{\otimes 3})$*, linearly spanned (over* $\mathrm{Hol}(\mathbb{C}-\Gamma)$*) by the elements of the form*

$$\xi' = e^{-(1)}_{-\lambda-\hbar(h^{(2)}+h^{(3)})}[\eta_1] \cdots e^{-(1)}_{-\lambda-\hbar(h^{(2)}+h^{(3)})-2(n-1)\hbar}[\eta_n](1 \otimes a \otimes b), \tag{45}$$

$n \geq 0$ (recall that the empty product is equal to 1), where $\eta_i \in \mathcal{K}$, and $a, b \in A(\tau)$ are such that $[h^{(1)} + h^{(2)} + h^{(3)}, \xi'] = 0$; and $A^{\cdot,\cdot,+}$ as the subspace of $\mathrm{Hol}(\mathbb{C}-\Gamma, A(\tau)^{\otimes 3})$ *spanned (over* $\mathrm{Hol}(\mathbb{C}-\Gamma)$*) by the elements of the form*

$$\eta' = (a' \otimes b' \otimes 1) f^{(3)}[r_1] \cdots f^{(3)}[r_n](h^{(3)})^s, \quad n, s \geq 0,$$

where $a', b' \in A(\tau)$, $r_i \in \mathcal{O}$, and such that $[h^{(1)} + h^{(2)} + h^{(3)}, \eta'] = 0$.

Definition 4.3. *$A^{+,\cdot,\cdot}$ is the subspace of the algebra* $\mathrm{Hol}(\mathbb{C}-\Gamma, A(\tau)^{\otimes 3})$ *linearly spanned (over* $\mathrm{Hol}(\mathbb{C}-\Gamma)$*) by the elements of the form*

$$\xi' = e^{(1)}[r_1] \dots e^{(1)}[r_n](1 \otimes a \otimes b), \quad n \geq 0, \tag{46}$$

where $r_i \in \mathcal{O}$, and $a, b \in A(\tau)$ are such that $[h^{(1)} + h^{(2)} + h^{(3)}, \xi'] = 0$.

$A^{\cdot,\cdot,-}$ is the subspace of $\mathrm{Hol}(\mathbb{C}-\Gamma, A(\tau)^{\otimes 3})$ *linearly spanned (over* $\mathrm{Hol}(\mathbb{C}-\Gamma)$*) by the elements of the form*

$$\eta' = (a' \otimes b' \otimes 1) f^{-(3)}_{\lambda+\hbar h^{(3)}-2\hbar}[\eta_1] \dots f^{-(3)}_{\lambda+\hbar h^{(3)}-2\hbar}[\eta_n](h^{(3)})^s, \quad n, s \geq 0, \tag{47}$$

where $\eta_i \in \mathcal{K}$, and $a', b' \in A(\tau)$ are such that $[h^{(1)} + h^{(2)} + h^{(3)}, \eta'] = 0$.

Proposition 4.1. *$A^{-,\cdot,\cdot}$, $A^{\cdot,\cdot,+}$, $A^{+,\cdot,\cdot}$ and $A^{\cdot,\cdot,-}$ are subalgebras of* $\mathrm{Hol}(\mathbb{C}-\Gamma, A(\tau)^{\otimes 3})$. *We have*

$$(\Delta \otimes 1)(A^{-+}) \subset A^{-,\cdot,\cdot} \cap A^{\cdot,\cdot,+}, \quad (1 \otimes \Delta)(A^{-+}) \subset A^{-,\cdot,\cdot} \cap A^{\cdot,\cdot,+}, \tag{48}$$

$$(\bar{\Delta} \otimes 1)(A^{+-}) \subset A^{+,\cdot,\cdot} \cap A^{\cdot,\cdot,-}, \quad (1 \otimes \bar{\Delta})(A^{+-}) \subset A^{+,\cdot,\cdot} \cap A^{\cdot,\cdot,-}. \tag{49}$$

The first statement is a consequence of the following relations between half-currents:

Lemma 4.1. *The generating series $e_\lambda^\pm(z), f_\lambda^\pm(z)$ satisfy the following relations:*

$$\frac{\theta(z-w-\hbar)}{\theta(z-w)} e^\epsilon_{\lambda+\hbar}(z) e^{\epsilon'}_{\lambda-\hbar}(w) + \epsilon\epsilon' \frac{\theta(w-z-\lambda)\theta(-\hbar)}{\theta(w-z)\theta(-\lambda)} e^{\epsilon'}_{\lambda+\hbar}(w) e^{\epsilon'}_{\lambda-\hbar}(w) \tag{50}$$
$$= \frac{\theta(z-w+\hbar)}{\theta(z-w)} e^{\epsilon'}_{\lambda+\hbar}(w) e^{\epsilon}_{\lambda-\hbar}(z) + \epsilon\epsilon' \frac{\theta(z-w-\lambda)\theta(-\hbar)}{\theta(z-w)\theta(-\lambda)} e^{\epsilon}_{\lambda+\hbar}(z) e^{\epsilon}_{\lambda-\hbar}(z),$$

$$\frac{\theta(z-w+\hbar)}{\theta(z-w)} f^\epsilon_{\lambda-\hbar}(z) f^{\epsilon'}_{\lambda+\hbar}(w) + \epsilon\epsilon' \frac{\theta(w-z-\lambda)\theta(\hbar)}{\theta(w-z)\theta(-\lambda)} f^{\epsilon'}_{\lambda-\hbar}(w) f^{\epsilon'}_{\lambda+\hbar}(w) \tag{51}$$
$$= \frac{\theta(z-w-\hbar)}{\theta(z-w)} f^{\epsilon'}_{\lambda-\hbar}(w) f^{\epsilon}_{\lambda+\hbar}(z) + \epsilon\epsilon' \frac{\theta(z-w-\lambda)\theta(\hbar)}{\theta(z-w)\theta(-\lambda)} f^{\epsilon}_{\lambda-\hbar}(z) f^{\epsilon}_{\lambda+\hbar}(z),$$

where ϵ, ϵ' take the values $+, -$.

To prove the statement about coproducts in Prop. 4.1, it is then sufficient to prove it from generators of A^{+-} and A^{-+} (see [9], Lemmas 1.2-4).

A useful argument in sect. 4.1 was that the intersection of $A^{\geq 0} \otimes A \otimes A^{<0}$ and $A^{<0} \otimes A \otimes A^{\geq 0}$ is reduced to $1 \otimes A \otimes 1$. Similarly, we have:

Proposition 4.2. *We have*

$$A^{+,\cdot,\cdot} \cap A^{-,\cdot,\cdot} = \mathrm{Hol}(\mathbb{C}-\Gamma, 1 \otimes (A(\tau)^{\otimes 2})^{\mathfrak{h}}), \ A^{\cdot,\cdot,+} \cap A^{\cdot,\cdot,-} = \mathrm{Hol}(\mathbb{C}-\Gamma, (A(\tau)^{\otimes 2})^{\mathfrak{h}} \otimes \mathbb{C}[h]),$$

where $(A(\tau)^{\otimes 2})^{\mathfrak{h}}$ are the elements of $A(\tau)^{\otimes 2}$ commuting with $h^{(1)} + h^{(2)}$.

4.3. **Decomposition of F.** We then have:

Proposition 4.3. *There is a unique a decomposition of F as*

$$F = F^2_\lambda F^1_\lambda, \quad \text{with} \quad F^1_\lambda \in A^{-+} \quad \text{and} \quad F^2_\lambda \in A^{+-}, \tag{52}$$

with $(\varepsilon \otimes 1)(F_i) = (1 \otimes \varepsilon)(F_i) = 1$, $i = 1, 2$.

As in the rational case, the proof of this fact relies on a study of the Hopf duality between the opposite quantum nilpotent current subalgebras of $U_\hbar \mathfrak{g}(\tau)$.

Lemma 4.2. *We have an expansion*

$$F^1_\lambda \in 1 + \hbar \sum_{i \geq 0} e^{-(1)}_{-\lambda - \hbar h^{(2)}}[e_{i;-\lambda}] f^{(2)}[e^i] + U_\hbar \mathfrak{n}_+(\tau)^{\geq 2} \otimes U_\hbar \mathfrak{n}_-(\tau)^{\geq 2} \mathbb{C}[h],$$

where $U_\hbar\mathfrak{n}_\pm(\tau)^{\geq 2} = \oplus_{i\geq 2} U_\hbar\mathfrak{n}_\pm(\tau)^{[i]}$*, and* $U_\hbar\mathfrak{n}_\pm(\tau)^{[i]}$ *is the linear span of the products of* i *terms of the form* $e[\epsilon]$ *in* $\pm = +$*, and* $f[\epsilon]$ *if* $\pm = -$.

4.4. **Twisted cocycle equation.**

Proposition 4.4. *The family* $(F^1_\lambda)_{\lambda\in\mathbb{C}-\Gamma}$ *satisfies the twisted cocycle equation*

$$F^{1(12)}_{\lambda+\hbar h^{(3)}}(\Delta\otimes 1)(F^1_\lambda) = F^{1(23)}_\lambda(1\otimes\Delta)(F^1_\lambda). \tag{53}$$

To show this result, we proceed as follows. Define for $\lambda\in\mathbb{C}-\Gamma$,

$$\Phi_\lambda = F^{1(12)}_{\lambda+\hbar h^{(3)}}(\Delta\otimes 1)(F^1_\lambda)\left(F^{1(23)}_\lambda(1\otimes\Delta)(F^1_\lambda)\right)^{-1}.$$

Then (48) implies that $(\Phi_\lambda)_{\lambda\in\mathbb{C}-\Gamma}$ belongs to $A^{-,\cdot,\cdot}\cap A^{\cdot,\cdot,+}$. On the other hand, using (33), we may rewrite Φ_λ as

$$\Phi_\lambda = \left((\bar\Delta\otimes 1)(F^2_\lambda)(F^{2(12)}_{\lambda+\hbar h^{(3)}})\right)^{-1}(1\otimes\bar\Delta)(F^2_\lambda)(1\otimes F^{2(23)}_\lambda).$$

Then (49) then implies that $(\Phi_\lambda)_{\lambda\in\mathbb{C}-\Gamma}$ belongs to $A^{+,\cdot,\cdot}\cap A^{\cdot,\cdot,-}$.

Prop. 4.2 then implies that

$$\Phi_\lambda = \sum_{i\geq 0} 1\otimes a^{(i)}_\lambda\otimes h^i. \tag{54}$$

Define now for $x\in A(\tau)$,

$$\Delta_\lambda(x) = F^1_\lambda\Delta(x)(F^1_\lambda)^{-1};$$

$(\Delta_\lambda)_{\lambda\in\mathbb{C}-\Gamma}$ and $(\Phi_\lambda)_{\lambda\in\mathbb{C}-\Gamma}$ satisfy the compatibility condition

$$\begin{aligned}&(\Delta_{\lambda+\hbar(h^{(3)}+h^{(4)})}\otimes 1\otimes 1)(\Phi_\lambda)(1\otimes 1\otimes\Delta_\lambda)(\Phi_\lambda)\\ &=\Phi^{(123)}_{\lambda+\hbar h^{(4)}}(1\otimes\Delta_{\lambda+\hbar h^{(4)}}\otimes 1)(\Phi_\lambda)\Phi^{(234)}_\lambda.\end{aligned} \tag{55}$$

It follows that $\Phi_\lambda = 1$.

4.5. **The BBB result.** The paper [3], sect. 2, contains the following result:

Proposition 4.5. *(see* [3]*) Let* $(\mathcal{A},\Delta^{\mathcal{A}}_\infty,\mathcal{R}^{\mathcal{A}}_\infty)$ *be a quasi-triangular Hopf algebra, with a fixed element* $\widetilde{h}$*. Let* $F(\lambda)$ *be a family of invertible elements of* $\mathcal{A}\otimes\mathcal{A}$*, parametrized by some subset* $U\subset\mathbb{C}$*. Set* $\Delta(\lambda) = \mathrm{Ad}(F(\lambda))\circ\Delta^{\mathcal{A}}_\infty$*. Suppose that the identity*

$$F^{(12)}(\lambda+\hbar\widetilde{h}^{(3)})(\Delta^{\mathcal{A}}_\infty\otimes 1)(F(\lambda)) = F^{(23)}(\lambda)(1\otimes\Delta^{\mathcal{A}}_\infty)(F(\lambda)) \tag{56}$$

is satisfied. Then we have

$$(\Delta(\lambda+\hbar\widetilde{h}^{(3)})\otimes 1)\circ\Delta(\lambda) = (1\otimes\Delta(\lambda))\circ\Delta(\lambda), \tag{57}$$

and if we set $\mathcal{R}(\lambda) = F^{(21)}(\lambda)\mathcal{R}_\infty^{\mathcal{A}} F(\lambda)^{-1}$*, we have the identity*

$$\mathcal{R}^{(12)}(\lambda)\mathcal{R}^{(13)}(\lambda+\hbar\widetilde{h}^{(2)})\mathcal{R}^{(23)}(\lambda) = \mathcal{R}^{(23)}(\lambda+\hbar\widetilde{h}^{(1)})\mathcal{R}^{(13)}(\lambda)\mathcal{R}^{(12)}(\lambda+\hbar\widetilde{h}^{(3)}). \tag{58}$$

Equation (58) is the algebraic version of the DYBE.

4.6. **End of the proof.** Apply now Prop. 4.5 to $\mathcal{A} = U_\hbar\mathfrak{g}(\tau)$, $\Delta_\infty^{\mathcal{A}} = \Delta$, $\mathcal{R}_\infty^{\mathcal{A}} = \mathcal{R}_\infty$ and $F(\lambda) = F_\lambda^1$. Set $\mathcal{R}_\lambda = (F_\lambda^1)^{(21)}\mathcal{R}_\infty(F_\lambda^1)^{-1}$. We have:

Corollary 4.1. *The family* $(\mathcal{R}_\lambda)_{\lambda\in\mathbb{C}-\Gamma}$ *satisfies the dynamical Yang-Baxter relation*

$$\mathcal{R}_\lambda^{(12)}\mathcal{R}_{\lambda+\hbar h^{(2)}}^{(13)}\mathcal{R}_\lambda^{(23)} = \mathcal{R}_{\lambda+\hbar h^{(1)}}^{(23)}\mathcal{R}_\lambda^{(13)}\mathcal{R}_{\lambda+\hbar h^{(3)}}^{(12)}. \tag{59}$$

The procedure to derive RLL relations from a Yang-Baxter-like equation is to first study finite-dimensional representations of the algebra, and then take the image of the YBE in these representations (see [22]).

Proposition 4.6. *(see* [11]*, Prop. 9) There is a morphism of algebras* $\pi_\zeta : A(\tau) \to \mathrm{End}(\mathbb{C}^2)\otimes\mathcal{K}_\zeta[\partial_\zeta][[\hbar]]$*, defined by the formulas*

$$\pi_\zeta(K) = 0, \quad \pi_\zeta(D) = \mathrm{Id}_{\mathbb{C}^2}\otimes\partial_\zeta,$$

$$\pi_\zeta(h[r]) = E_{11}\otimes\left(\frac{2}{1+q^{\partial}}r\right)(\zeta) - E_{-1-1}\otimes\left(\frac{2}{1+q^{-\partial}}r\right)(\zeta), \quad r\in\mathcal{O},$$

$$\pi_\zeta(h[\lambda]) = E_{11}\otimes\left(\frac{1-q^{-\partial}}{\hbar\partial}\lambda\right)(\zeta) - E_{-1-1}\otimes\left(\frac{q^{\partial}-1}{\hbar\partial}\lambda\right)(\zeta), \quad \lambda\in L_0,$$

$$\pi_\zeta(e[\epsilon]) = \frac{\theta(\hbar)}{\hbar}E_{1,-1}\otimes\epsilon(\zeta), \quad \pi_\zeta(f[\epsilon]) = E_{-1,1}\otimes\epsilon(\zeta), \quad \epsilon\in\mathcal{K}.$$

The image of $\mathcal{R}_\lambda$ by these representations is computed as follows:

Lemma 4.3. *The image of* $\mathcal{R}_\lambda$ *by* $\pi_\zeta\otimes\pi_{\zeta'}$ *is*

$$(\pi_\zeta\otimes\pi_{\zeta'})(\mathcal{R}_{\lambda+\hbar}) = A(\zeta,\zeta')R^-(\zeta-\zeta',\lambda), \tag{60}$$

where $R^-(z,\lambda)$ *has been defined in (41).*

The computation relies on Lemma 4.2.

Appling $id\otimes\pi_\zeta\otimes\pi_{\zeta'}$, $\pi_\zeta\otimes\pi_{\zeta'}\otimes id$ and $\pi_\zeta\otimes id\otimes\pi_{\zeta'}$ to (59), after the change of λ into $\lambda+\hbar$, we find (42) and (43).

References

[1] G.E. Arutyunov, L.O. Chekhov, S.A. Frolov, R-matrix quantization of the elliptic Ruijsenaars-Schneider model, q-alg/9612032.

[2] J. Avan, O. Babelon, E. Billey, The Gervais-Neveu-Felder equation and quantum Calogero-Moser system, Comm. Math. Phys. 178 (1996), 281-99.

[3] O. Babelon, D. Bernard, E. Billey, A quasi-Hopf algebra interpretation of quantum $3j$ and $6j$ symbols and difference equations, q-alg/9511019, Phys. Lett. B. 375 (1996), 89-97.

[4] J. Ding, I.B. Frenkel, Isomorphism of two realizations of quantum affine algebras $U_q(\hat{\mathfrak{gl}}_n)$, Comm. Math. Phys. 156 (1993), 277-300.

[5] J. Ding, K. Iohara, Generalization and deformation of Drinfeld quantum affine algebras, preprint q-alg/9608002.

[6] V.G. Drinfeld, A new realization of Yangians and quantized affine algebras, Sov. Math. Dokl. 36 (1988).

[7] V.G. Drinfeld, Quasi-Hopf algebras, Leningrad Math. J. 1:6 (1990), 1419-57.

[8] B. Enriquez, G. Felder, A construction of Hopf algebra cocycles for double Yangians, preprint q-alg/9703012.

[9] B. Enriquez, G. Felder, Elliptic quantum groups $E_{\tau,\eta}(\mathfrak{sl}_2)$ ans quasi-Hopf algebras, preprint q-alg/9703018.

[10] B. Enriquez, V.N. Rubtsov, Hitchin systems, higher Gaudin operators and r-matrices, Math. Res. Lett. 3 (1996), 343-57.

[11] B. Enriquez, V.N. Rubtsov, Quantum groups in higher genus and Drinfeld's new realizations method ($\mathfrak{sl}_2$ case), preprint Ecole Polytechnique, no. 1123, q-alg/9601022.

[12] B. Enriquez, V.N. Rubtsov, Quasi-Hopf algebras associated with $\mathfrak{sl}_2$ and complex curves, preprint Ecole Polytechnique, no. 1145, q-alg/9608005.

[13] L. Faddeev, N. Reshetikhin, L. Takhtajan, Quantization of Lie groups and Lie algebras, Leningrad Math. J. 1 (1989), 178-201.

[14] G. Felder, Conformal field theory and integrable systems associated to elliptic curves, Proc. ICM Zürich 1994, 1247-55, Birkhäuser (1994); Elliptic quantum groups, Proc. ICMP Paris 1994, 211-8, International Press (1995).

[15] G. Felder, V. Tarasov, A. Varchenko, Solutions of the QKZB equations and Bethe Ansatz I, preprint q-alg/9606005.

[16] G. Felder, A. Varchenko, On representations of the elliptic quantum group $E_{\tau,\eta}(\mathfrak{sl}_2)$, q-alg/9601003; Comm. Math. Phys. 181 (1996), 741-61.

[17] G. Felder, C. Wieczerkowski, Conformal field theory on elliptic curves and Knizhnik-Zamolod-chikov-Bernard equations, hep-th/9411004, Comm. Math. Phys. 176 (1996), 133-62.

[18] E. Frenkel, N. Reshetikhin, Quantum affine algebras and deformations of the Virasoro and $\mathcal{W}$-algebras, q-alg/9505025.

[19] O. Foda, K. Iohara, M. Jimbo, T. Miwa, H. Yan, An elliptic algebra for $\mathfrak{sl}_2$, preprint RIMS 974.
[20] S. Khoroshkin, Central extension of the Yangian double, preprint q-alg/9602031.
[21] S. Khoroshkin, V. Tolstoy, Yangian double, Lett. Math. Phys., to appear.
[22] N. Reshetikhin, M. Semenov-Tian-Shansky, Central extensions of quantum current groups, Lett. Math. Phys. 19 (1990), 133-42.

Centre de Mathématiques, URA 169 du CNRS, Ecole Polytechnique, 91128 Palai-seau, France